https://www.linkedin.com/feed/update/urn:li:activity:6480681711180095488

The two sequences (a_n) and (b_n) are defined by

$a_n = \sum_{k=1}^{n} \ln \frac{2k-1}{2k}$ and $2b_n = \ln(2n+1) - a_n, n \geq 1.$

Prove that the two sequences are strictly monotonic and unbounded.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let $c_n := \sum_{k=1}^{n} \ln \frac{2k}{2k-1}$. Then $c_{n+1} - c_n = \ln \frac{2n+1}{2n} = \ln\left(1 + \frac{1}{2n}\right) > \frac{1}{2n+1}$ for any $n \in \mathbb{N}$. ($\left(1+\frac{1}{m}\right)^{m+1} > e, \forall m \in \mathbb{N}$ implies $\ln\left(1+\frac{1}{m}\right) > \frac{1}{m+1}$, $\forall m \in \mathbb{N}$).

Hence, $c_{n+1} > c_n, \forall n \in \mathbb{N}$ and $c_n - c_1 = \sum_{k=1}^{n} \frac{1}{2k+1} > \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k+1} = \frac{1}{2}(h_{n+1} - 1),$

where $h_n = \sum_{k=1}^{n} \frac{1}{k}$ is n-th harmonic number.

Since $h_n > \ln(n+1)$ (because $\left(1+\frac{1}{n}\right)^n < e \Leftrightarrow \ln(n+1) - \ln n < \frac{1}{n}, \forall n \in \mathbb{N}$ implies $\sum_{k=1}^{n}(\ln(k+1) - \ln k) < \sum_{k=1}^{n} \frac{1}{k} = h_n \Leftrightarrow \ln(n+1) < h_n, \forall n \in \mathbb{N}$) then $c_n > \ln 2 + \frac{1}{2}(h_{n+1} - 1) > \ln 2 + \frac{1}{2}(\ln(n+2) - 1)$. Thus, (c_n) is strictly increasing and unbounded from above sequence and, therefore, sequence $(a_n) = (-c_n)$ is strictly decreasing and unbounded from below.

Since $b_n = \frac{1}{2}(\ln(2n+1) - a_n) = \frac{1}{2}(\ln(2n+1) + c_n)$ then (b_n) is strictly increasing and unbounded from above sequence as sum of two strictly increasing and unbounded from above sequences.